

# ON EXTENSIONS OF THE TORELLI MAP

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**ABSTRACT.** The divisors on  $\overline{M}_g$  that arise as the pullbacks of ample divisors along any extension of the Torelli map to any toroidal compactification of  $A_g$  form a 2-dimensional extremal face of the nef cone of  $\overline{M}_g$ , which is explicitly described.

## 1. INTRODUCTION

The moduli spaces  $M_g$ , of smooth curves of genus  $g$ , and  $A_g$ , of  $g$ -dimensional principally polarized Abelian varieties are related by a natural embedding  $t_g : M_g \rightarrow A_g$ , that sends the isomorphism class of a smooth genus  $g$  curve  $C$  to a pair consisting of the isomorphism class of its Jacobian together with its theta divisor  $\Theta_{g-1}$ . The map is named for Torelli who proved that it is injective [Wei57].

Both  $M_g$  and  $A_g$  are quasi-projective varieties. The Deligne-Mumford compactification  $\overline{M}_g$  is obtained by adding nodal curves having a finite number of automorphisms. While this is often taken to be the best compactification of  $M_g$ , the closure in any compactification of  $A_g$  of the image of  $M_g$  under the Torelli map gives an alternative one. In particular, if the Torelli map extends to a morphism from  $\overline{M}_g$  to that compactification of  $A_g$ , then one may be able to say something interesting about the moduli space of curves. For example, the Satake compactification  $A_g^{Sat}$  is the normal projective variety constructed by taking Proj of the graded ring of Siegel modular forms [Sat60], [FC90]. The Torelli map extends to a morphism  $f_\lambda : \overline{M}_g \rightarrow \overline{A}_g^{Sat}$ , and by studying the image  $\overline{M}_g^{Sat}$  in  $A_g^{Sat}$ , one can prove the very interesting and nontrivial fact that  $M_g$  contains a complete curve for  $g \geq 3$  [Oor74].

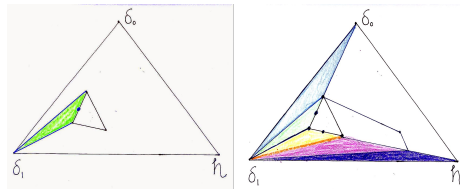
There are infinitely many toroidal compactifications  $A_g^\tau$  of  $A_g$ , each dependent on a choice of fan  $\tau$ , which describes a decomposition of the cone of real positive quadratic forms in  $g$  variables ([AMRT75], [FC90], [Nam80], [Ols08]). Each compactification  $A_g^\tau$  contains a common sublocus,  $A_g^{part}$ , the partial compactification of  $A_g$  defined by Mumford. Also, for each  $\tau$ , there is a morphism  $\eta_\tau$  from  $A_g^\tau$  to Satake's compactification  $A_g^{Sat}$ . The first result of this short note is that any regular extension of the Torelli map from  $\overline{M}_g$  to a compactification of  $A_g$  having these properties is given by a sublinear system of a divisor lying on what will be shown to be a 2-dimensional extremal face of the nef cone of  $\overline{M}_g$ :

**Theorem 1.1.** *Let  $X$  be any compactification of  $A_g$  that contains Mumford's partial compactification  $A_g^{part}$  and maps to Satake's compactification  $A_g^{Sat}$ . Then if  $f : \overline{M}_g \rightarrow X$  is any extension of the Torelli map and  $A$  any ample divisor on  $X$ , then  $f^*(A)$  lies in the interior of the face  $\mathbb{F} = \{\alpha\lambda + \beta(12\lambda - \delta_0) \mid \alpha, \beta \geq 0\}$ .*

Mumford and Namikawa [Nam76], have shown that  $t_g$  extends to a morphism  $f_{vor} : \overline{M}_g \rightarrow A_g^{vor}$  to the Voronoi compactification of  $A_g$ , and very recently Alexeev and Brunsyate [AB11] have shown that the Torelli map extends to a morphism  $f_{per} : \overline{M}_g \rightarrow \overline{A}_g^{per}$ , to the perfect cone compactification. By using the morphism  $f_{per}$ , it is possible to see that every divisor in  $\mathbb{F}$  gives rise to such an extension of the Torelli map:

**Theorem 1.2.** *Let  $f_{per} : \overline{M}_g \rightarrow \overline{A}_g^{per}$  be the Toroidal extension of the Torelli map defined in [AB11]. Then*

$$\mathbb{F} = f_{per}^*(\text{Nef}(A_g^{per})).$$



In particular, Theorem 1.2 shows that all divisors in the interior of  $\mathbb{F}$  are semi-ample. As is proved in Corollary 4.3, it follows from [SB06] that if we take  $\overline{M}_g$  to be defined over  $\mathbb{C}$ , then the nef divisor  $12\lambda - \delta_0$  is actually semi-ample for  $g \leq 11$ . To exhibit the nontriviality of these statements, in Remark 3.3 we explain that one cannot show that the divisors on  $\mathbb{F}$  are semi-ample by simply applying the base point free theorem.

$A_g^{per}$ ,  $A_g^{vor}$ , and  $A_g^{cen}$  have been the most extensively studied toroidal compactifications. Here  $\tau = cen$  denotes the central cone decomposition which was constructed as a blowup of  $A_g^{Sat}$  by Igusa in [Igu67]. In [Ale02], Alexeev has shown that  $A_g^{vor}$  is the normalization of the main irreducible component of the moduli space  $\overline{AP}_g$  parametrizing isomorphism classes  $(X, \theta)$  of stable semi-abelic pairs. Shepherd-Barron has shown that for  $g \geq 12$ , one has that  $A_g^{per}$  is the canonical model for the Satake compactification  $A_g^{Sat}$  [SB06], and of any smooth compactification of  $A_g$  (even when regarded as a stack over  $A_g^{Sat}$ ). Shepherd-Barron has remarked in [SB06] that he has no reason to believe that  $A_g^{per}$  is a moduli space; and no other Toroidal compactification is known to be such.

General relationships between the toroidal compactifications given by different fans are not fully understood. There is a rational map  $g : A_g^{vor} \dashrightarrow A_g^{per}$  which is an isomorphism for  $g = 2, 3$ , a morphism for  $g \leq 5$ , but not regular for  $g \geq 6$  ([AB11], [ER02], [ER01]). For  $g = 2$  and  $g = 3$  all three compactifications coincide, for  $g = 4$ , one has that  $A_4^{per} = A_4^{cen}$ , but are not equal to  $A_4^{vor}$ . For  $g \leq 7$ , that there is toroidal extension of the Torelli map  $f_{cen} : \overline{M}_g \rightarrow A_g^{cen}$ , but that there is no such extension for  $g \geq 9$  (cf. [VRG10], [AB11]). Although the Picard group of  $A_4^{vor}$  has been found [HS04] (it has 3 generators), even the number of boundary components of  $A_g^{vor}$  is unknown for general  $g$ . In contrast, the Picard group of  $A_g^{per}$  has rank 2, and the nef cone of  $A_g^{per}$  has been explicitly described [SB06].

The result from Theorem 1.1, that all pullbacks of ample divisors from toroidal compactifications lie in the face  $\mathbb{F}$ , begs the question of whether, in fact, the images of  $\overline{M}_g$  under these extensions are all isomorphic. There is some evidence for this possibility already. Alexeev and Bruniyate [AB11] show that the image of  $\overline{M}_g$  under  $f_{vor}$  is isomorphic to its image under their extension of Torelli map to the perfect compactification. One consequence of Theorem 1.1 gives at least a rough comparison of the images of  $\overline{M}_g$  under morphisms to them:

**Corollary 1.3.** *The image of  $\overline{M}_g$  under any extension of the Torelli map to a compactification of  $A_g$  that contains Mumford's partial compactification  $A_g^{part}$  and maps to Satake's compactification  $A_g^{Sat}$  has Picard number 2.*

The face  $\mathbb{F}$  is really the first region of the nef cone of  $\overline{M}_g$  to have been studied, and in fact, for  $g = 2$ , one has that  $\mathbb{F} = \text{Nef}(\overline{M}_2)$ . In [Fab90, Prop. 3.3], Faber used [CH88, Thm. 1.3] to establish a base case of an induction to show  $12\lambda - \delta_0$  is nef for  $g \geq 2$ . The divisor  $\lambda$  is semi-ample – and responsible for the morphism from  $\overline{M}_g$  to  $A_g^{Sat}$ . For  $g = 3$ , a cross-section of the corresponding chamber  $Ch(\mathbb{F})$  of the effective cone adjacent to  $\mathbb{F}$  is depicted on the left in the figure above. The image on the right shows the chambers of the effective cone that are known for  $\overline{M}_3$ . For general  $g \geq 2$ , the chamber  $Ch(\lambda)$ , directly adjacent to  $Ch(\mathbb{F})$ , corresponds to the morphism  $f_{Sat}$ . For  $g \geq 3$  the chambers on the other side of  $Ch(\mathbb{F})$  have been unearthed by the Hassett-Keel program ([Has05], [HH08], [HH09], [HL10], [HL09], [HL07]).

*Layout of the paper:* Background on divisors and curves on  $\overline{M}_g$  is given in Section 2. A combinatorial proof that  $\mathbb{F}$  is a two dimensional extremal face of the nef cone of  $\overline{M}_g$  is given in Section 3. The face  $\mathbb{F}$  is shown to be equal to the pullback of the nef cone of  $A_g^{per}$  along the newly discovered morphism  $f_{per}$  in Section 4. Finally, the main result, Theorem 1.1, that any possible extension of the Torelli map is given by a divisor on  $\mathbb{F}$  is proved in Section 5.

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## 2. BACKGROUND ON DIVISORS AND CURVES ON $\overline{M}_g$

As is conventional, let  $\lambda$  be the first Chern class of the Hodge bundle, and for  $0 \leq i \leq \lfloor \frac{g}{2} \rfloor$ , let  $\delta_i$  be the class of the boundary component  $\Delta_i$ . The divisor  $\Delta_0$  is the component of the boundary of  $\overline{M}_g$  whose generic element has a single node which is nonseparating, while for  $1 \leq i \leq \lfloor \frac{g}{2} \rfloor$ , the divisor  $\Delta_i$  is the component of the boundary of  $\overline{M}_g$  whose generic element has a single node separating the curve into a component of genus  $i$  and a component of genus  $g - i$ . For  $g \geq 3$ , the divisor classes in the set  $\{\lambda\} \cup \{\delta_i : 0 \leq i \leq \lfloor \frac{g}{2} \rfloor\}$  form a basis for  $\text{Pic}(\overline{M}_g)$ .

**Definition 2.1.** We will next recall the definition of the six types of curves on  $\overline{M}_g$ , which in the literature are called  $F$ -curves. The first curve  $C_1$  is a family of elliptic tails. The remaining five types are obtained by attaching curves to a fixed 4-pointed stable curve  $B$  of genus zero with one point moving and the other three fixed. The curves to attach are described as follows:

$C$	curves to attach to $B$ to obtain $C$
$C_2$	a 4-ptd curve of genus $g - 3$
$C_3^i$	a 1-ptd curve of genus $i$ and a 3-ptd curve of genus $g - 2 - i$
$C_4^i$	2-ptd curves of genus $i$ and of genus $g - 2 - i$
$C_5^{ij}$	1-ptd curves of genus $i$ and $j$ , a 2-ptd curve of genus $g - 1 - i - j$
$C_6^{ijk} = F_{i,j,k,\ell}$	1-ptd curves of genus $i, j, k$ , and $\ell = g - (i + j + k)$

In the table above, to define  $C_3^i$ , take  $1 \leq i \leq g - 2$ ; to define  $C_4^i$ , take  $0 \leq i \leq g - 2$ ; to define  $C_5^{ij}$ , take  $i$  and  $j \geq 1$ , such that  $i + j \leq g - 1$ ; and to define  $C_6^{ijk} = F_{i,j,k,\ell}$ , take  $i, j, k, \ell \geq 1$ . When attaching a 2-pointed curve of genus zero, replace the resulting rational bridge with a node.

The table below shows the intersection number for an arbitrary divisor  $D = a\lambda - \sum_{i=0}^{\lfloor \frac{g}{2} \rfloor} b_i \delta_i$  with the  $F$ -curves, and the degree of intersection for the divisors  $\lambda$  and  $12\lambda - \delta_0$ , which are the generators of the face  $\mathbb{F}$ .

$C$	$D \cdot C = (a\lambda - \sum_{i=0}^{\lfloor \frac{g}{2} \rfloor} b_i \delta_i) \cdot C$	$\lambda \cdot C$	$(12\lambda - \delta_0) \cdot C$
$C_1$	$\frac{a}{12} - b_0 + \frac{b_1}{12}$	$\frac{1}{12}$	0
$C_2$	$b_0$	0	1
$C_3^i$	$b_i$	0	0
$C_4^i$	$2b_0 - b_{i+1}$	0	2
$C_5^{ij}$	$b_i + b_j - b_{i+j}$	0	0
$C_6^{ijk} = F_{i,j,k,\ell}$	$b_i + b_j + b_k + b_\ell - b_{i+j} - b_{i+k} - b_{i+\ell}$	0	0

### 3. AN EXTREMAL FACE OF THE NEF CONE OF $\overline{M}_g$

**Proposition 3.1.** *The divisors  $\lambda$  and  $12\lambda - \delta_0$  generate independent extremal rays of  $\text{Nef}(\overline{M}_g)$ . The set of divisors*

$$\mathbb{F} = \{\alpha\lambda + \beta(12\lambda - \delta_0) : \alpha, \beta \geq 0\},$$

*is a 2-dimensional extremal face of  $\text{Nef}(\overline{M}_g)$ .*

To prove Proposition 3.1, we show that a particular set of  $F$ -curves is independent.

**Lemma 3.2.** *Let  $d$  be the dimension of the  $NS(\overline{M}_g)$ . The set  $S = \{C_1, C_2\} \cup \{C_3^i : 1 \leq i \leq d-2\}$  consists of  $d$  independent curves.*

*Proof.* (of Lemma 3.2) To show that  $S$  is independent, using Table 2, we form the matrix of intersection numbers of these curves with the basis  $12\lambda, 12\lambda - \delta_0, \delta_1, \dots, \delta_{d-2}$  for  $\text{Pic}(\overline{M}_g)$ . Since this is the identity matrix, it shows that the curves in  $S$  form a basis for the 1 cycles of  $\overline{M}_g$ .

	$C_1$	$C_2$	$C_3^1$	$C_3^2$	$\dots$	$C_3^{d-2}$
$12\lambda$	1	0	0	0	$\dots$	0
$12\lambda - \delta_0$	0	1	0	0	$\dots$	0
$\delta_1$	0	0	1	0	$\dots$	0
$\delta_2$	0	0	0	1	$\dots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	
$\delta_{d-2}$	0	0	0	0	$\dots$	1

□

*Proof.* (of Proposition 3.1) By [Mum83b], and [Fab90, Prop3.3] the divisors  $\lambda$ , and  $12\lambda - \delta_0$  are nef. Let  $S = \{C_1, C_2\} \cup \{C_3^1, \dots, C_3^{d-2}\}$ . Using Table 2, one can see that the intersection numbers with the curves defined in Lemma 3.2 are as follows:

- (1)  $\lambda \cdot C_1 \neq 0$ , and  $\lambda \cdot C = 0$ , for all  $C \in S \setminus C_1$ ;
- (2)  $(12\lambda - \delta_0) \cdot C_2 \neq 0$ , and  $(12\lambda - \delta_0) \cdot C = 0$ , for all  $C \in S \setminus C_2$ ;
- (3) For  $\epsilon > 0$ ,  $(\epsilon\lambda + 12\lambda - \delta_0) \cdot C_1 \neq 0$ ,  $(\epsilon\lambda + 12\lambda - \delta_0) \cdot C_2 \neq 0$ , and  $(\epsilon\lambda + 12\lambda - \delta_0) \cdot C = 0$ , for all  $C \in S \setminus \{C_1, C_2\}$ ;

Since by Lemma 3.2, all of the curves are independent, it follows that  $\lambda$  and  $12\lambda - \delta_0$  generate extremal rays of the nef cone of  $\overline{M}_g$  while positive linear combinations of them lie on the interior of a 2-dimensional extremal face. □

We shall see in Section 4 that all elements on the face of divisors  $\mathbb{F}$  are pullbacks of nef divisors along the morphism  $f_{per} : \overline{M}_g \rightarrow A_g^{per}$ , and in particular, the divisors on the interior of the face are semi-ample. In this section we discuss the fact that we do not know of another way to show that any element of that face is semi-ample. In particular we make the following observation.

**Remark 3.3.** One cannot trivially use the basepoint-free Theorem to prove any divisor  $D_{\alpha\beta} = \alpha\lambda + \beta(12\lambda - \delta_0)$ , for  $\alpha$ , and  $\beta \geq 0$  is semi-ample. Indeed, recall that the basepoint-free Theorem [KM98, Thm. 3.3] says that if  $(X, \Delta)$  is a proper klt pair with  $\Delta$  effective, and if  $D$  is a nef Cartier divisor such that  $mD - K_X - \Delta$  is nef and big for some  $m > 0$ , then  $|kD|$  has no basepoints for all  $k \gg 0$ . By [BCHM10, Lemma 10.1], the pair  $(\overline{M}_g, \Delta)$ , with  $\Delta = \sum_{i=0}^{\lfloor \frac{g}{2} \rfloor} \delta_i$  on  $\overline{M}_g$  is klt. The canonical divisor of  $\overline{M}_g$  can be expressed as

$$K_{\overline{M}_g} = 13\lambda - 2 \sum_{i=0}^{\lfloor \frac{g}{2} \rfloor} \delta_i.$$

It is easy to see, no matter what choice of nonnegative  $\alpha$ , and  $\beta$ , that one can't find  $m$  so that  $mD_{\alpha\beta} - (K_{\overline{M}_g} + \Delta)$  is nef. Indeed,

$$mD_{\alpha\beta} - (K_{\overline{M}_g} + \Delta) = (m(\alpha + 12\beta) - 13)\lambda - (m\beta - 1)\delta_0 - \sum_{i=1}^{\lfloor \frac{g}{2} \rfloor} (-1)\delta_i.$$

If this divisor were nef, then it would be an  $F$ -divisor, which is a divisor that nonnegatively intersects all  $F$ -curves on  $\overline{M}_g$ . However, it intersects the curves  $C_3^i$  in degree  $-1$ .

#### 4. THE PULLBACK OF THE NEF CONE OF $A_g^{per}$

In this section we show that the two dimensional face  $\mathbb{F}$  spanned by the extremal divisors  $\lambda$  and  $12\lambda - \delta_0$  is equal to the pullback of the nef cone of  $A_g^{per}$  along the morphism  $f_{per} : \overline{M}_g \rightarrow A_g^{per}$ . This shows that in particular, the divisors on the interior of the face  $\mathbb{F}$  are semi-ample. As  $\lambda$  is the pullback of the ample divisor  $M$  of weight 1 modular forms on  $A_g^{Sat}$  along the morphism  $f_{Sat} : \overline{M}_g \rightarrow A_g^{Sat}$ , one has that  $\lambda$  is semi-ample. Ideally one would like to know that  $12\lambda - \delta_0$  is also semi-ample. Using Shepherd-Barron's results about semi-ampleness of the extremal divisor  $12M - D_g^{per}$  for low  $g$ , we show that for  $g \leq 11$ , as long as  $\overline{M}_g$  is defined over  $\mathbb{C}$ , then this is true.

We will use the following to prove Proposition 4.2 as well as to prove Theorem 1.1.

**Lemma 4.1.** *For  $j : M_g^* \hookrightarrow \overline{M}_g$  the embedding of the moduli space  $M_g^*$  of stable curves of genus  $g$  of compact type, one has that*

$$\text{Ker}(j^* : \text{Pic}(\overline{M}_g)_{\mathbb{Q}} \rightarrow \text{Pic}(M_g^*)_{\mathbb{Q}}) \cong \mathbb{Q}\delta_0.$$

*Proof.* Let  $X$  be any compactification of  $A_g$  for which there is a morphism  $\eta : X \rightarrow A_g^{Sat}$ , and such that  $\eta^{-1}(A_g \cup A_{g-1}) = A_g^{part} = A_g \cup D_g$  is Mumford's partial compactification of  $A_g$ . Suppose that  $f : \overline{M}_g \rightarrow X$  any morphism that extends the Torelli map. Note that  $f_{Sat} = \eta \circ f$ , and  $M_g^* = f_{Sat}^{-1}(A_g)$ . We have the following commutative diagram.

$$\begin{array}{ccccc} & & X & & \\ & \nearrow f & \uparrow i & \searrow \eta & \\ \overline{M}_g & \xleftarrow{j} M_g^* & \xrightarrow{\phi} A_g^{part} & \xrightarrow{\quad} A_g^{Sat} = A_g \cup A_{g-1} \cup \dots \cup A_0. \end{array}$$

The inclusion

$$\overline{M}_g \setminus \Delta_0 \xrightarrow{j} M_g^*,$$

induces an isomorphism

$$A^1(M_g^*) \cong A^1(\overline{M}_g \setminus \Delta_0),$$

giving the following right exact sequence

$$\mathbb{Z}\delta_0 \rightarrow A^1(\overline{M}_g) \rightarrow A^1(M_g^*) \cong A^1(\overline{M}_g \setminus \Delta_0) \rightarrow 0.$$

Tensoring with  $\mathbb{Q}$  is exact and so we get the right exact sequence

$$\mathbb{Q}\delta_0 \longrightarrow \text{Pic}(\overline{M}_g)_{\mathbb{Q}} \xrightarrow{j^*} \text{Pic}(M_g^*)_{\mathbb{Q}} \longrightarrow 0.$$

In particular, elements in the kernel of  $j^*$  are equivalent to rational multiples of  $\delta_0$ .  $\square$

**Proposition 4.2.** *Let  $f_{per} : \overline{M}_g \longrightarrow A_g^{per}$  be the Toroidal extension of the Torelli map. Then*

$$f_{per}^*(\text{Nef}(A_g^{per})) = \mathbb{F}$$

*Proof.* There is a morphism  $\eta_{per} : A_g^{per} \longrightarrow A_g^{Sat}$ , and that  $A_g^{per}$  contains  $\eta_{per}^{-1}(A_g \cup A_{g-1}) = A_g^{part} = A_g \cup D_g$ . By [SB06, Theorem 0.1],

$$\text{Nef}(A_g^{per}) = \{aM - bD_g^{per} : a \geq 12b \geq 0\},$$

where  $M = \eta_{per}^*(M)$ ,  $M$  is the ample generator of  $\text{Pic}(A_g^{Sat})_{\mathbb{Q}}$ , and  $D_g^{per}$  is the closure of  $D_g$  in  $A_g^{per}$ . In particular,  $12M - D_g^{per}$  and  $M$  generate the nef cone of  $A_g^{per}$ . The result will therefore follow after we show that  $f_{per}^*(M) = \lambda$  and that  $f_{per}^*(12M - D_g^{per}) = 12\lambda - \delta_0$ .

Consider the following commutative diagram.

$$\begin{array}{ccccc} & & & A_g^{per} & \\ & & f_{per} \nearrow & \uparrow i & \searrow \eta_{per} \\ \overline{M}_g & \xleftarrow{j} & M_g^* & \xrightarrow{\phi} & A_g^{part} = A_g \cup D_g & \longrightarrow & A_g^{Sat} \end{array}$$

Recall that  $f_{Sat}^{-1}(A_g) = M_g^*$  is the moduli space of genus  $g$  stable curves of compact type [MFK94]. We denote by  $M$  the pull back of  $M$  to the two varieties  $M_g^*$ , and  $A_g^{part}$ . By Lemma 4.1, any element of the kernel of  $j^*$  is equivalent to a rational multiple of  $\delta_0$ . In particular, if  $\lambda, \delta_0, B_3, \dots, B_d$  is any basis for  $\text{Pic}(\overline{M}_g)_{\mathbb{Q}}$ , then  $j^*\lambda, j^*B_3, \dots, j^*B_d$  is a basis for  $\text{Pic}(M_g^*)_{\mathbb{Q}}$ .

Now we are in the position to prove our assertion. Let  $\mathcal{D}$  be any nef divisor on  $A_g^{per}$ , and let  $\{\lambda, \delta_0, B_3, \dots, B_d\}$  is any basis for  $\text{Pic}(\overline{M}_g)_{\mathbb{Q}}$ . Then  $f_{per}^*(\mathcal{D}) = a\lambda - b\delta_0 - \sum_{i=3}^d b_i B_i$ , and  $j^*(f_{per}^*\mathcal{D}) = aj^*\lambda - \sum_{i=3}^d b_i j^*B_i$ . We will first show that the  $b_i = 0$  for  $i \geq 3$ . To do this we'll see that  $\sum_{i=3}^d b_i j^*B_i \in \ker(j^*)$ .

Since  $\mathcal{D}$  is nef on  $A_g^{per}$ , we may write  $\mathcal{D} = \alpha L - \beta D_g^{per}$  such that  $\alpha \geq 12\beta \geq 0$ . We also have that  $i^*(\mathcal{D}) = \alpha L - \beta D_g$ . On the other hand,  $\phi^*D_g = 0$ , since  $M_g^* = f_{Sat}^{-1}(A_g)$ . By commutativity of the diagram, it follows therefore that  $j^*f_{per}^*(\mathcal{D}) = \phi^*(i^*(\mathcal{D})) = \alpha L$ .

Comparing the coefficients of the basis elements:

$$aj^*\lambda - \sum_{i=3}^d b_i j^*B_i = \alpha L - \sum_{i=3}^d b_i j^*B_i = \alpha L.$$

And so we see that  $a = \alpha$  and  $b_i = 0$ , for  $3 \leq i \leq d$ . In particular, we have shown that  $f_{per}^*(L) = \lambda$  and that  $f_{per}^*(D_g^{per}) = c\delta_0$ , for some  $c \in \mathbb{Q}$ . We will show that  $c$  has to be 1.

In [SB06], Shepherd-Barron has shown that the Mori Cone of curves of  $A_g^{per}$  is generated by curves  $C(1)$  and  $C(2)$ , where  $C(1)$  is the closure of the set of points  $B \times E$  where  $B$  is a fixed principally polarized abelian  $(g-1)$ -fold and  $E$  is a variable elliptic curve, and  $C(2)$  is any exceptional curve of the contraction  $\eta_{per} : A_g^{per} \longrightarrow A_g^{Sat}$ . The image of the F-curve  $C_1$  on  $\overline{M}_g$ , defined in Definition 2.1, under  $f_{Sat}$  is the same as the image of  $C(1)$  under the map  $\eta_{per}$  which gives that  $C(1)$  is the image under  $f_{per}$  of the F-Curve  $C_1$ .

Shepherd-Barron showed that  $C(1) \cdot (12L - D_g^{per}) = 0$  and we have shown that  $C_1 \cdot \lambda \neq 0$ . If  $c = 0$ , then by the projection formula

$$\begin{aligned} 0 &= C(1) \cdot (12L - cD_g^{per}) = (f_{per})_*(C_1) \cdot (12L - cD_g) \\ &= (f_{per})_*(C_1 \cdot f_{per}^*(12L - cD_g)) = (f_{per})_*(C_1 \cdot f_{per}^*(12L)) = (f_{per})_*(C_1 \cdot 12\lambda) \neq 0, \end{aligned}$$

which is clearly a contradiction.

Therefore, assuming  $c \neq 0$ , we consider the divisor  $f_{per}^*(12L - \frac{1}{c}D_g^{per}) = 12\lambda - \delta_0$ .

$$0 = C_1 \cdot (12\lambda - \delta_0) = C_1 \cdot f_{per}^*(12L - \frac{1}{c}D_g^{per}) = (f_{per})_*(C_1) \cdot (12L - \frac{1}{c}D_g^{per}).$$

But  $(f_{per})_*(C_1) = C(1)$  and the kernel of  $C(1)$  is a line in the 2-dimensional vector space  $NS(A_g^{per})_{\mathbb{Q}}$  that contains the point  $12L - D_g^{per}$ . So in other words,  $12L - \frac{1}{c}D_g^{per}$  must be a rational multiple of  $12L - D_g^{per}$ . This can only happen if  $c = 1$ . We have shown that  $f_{per}^*L = \lambda$  and that  $f_{per}^*(12L - D_g^{vor}) = 12\lambda - \delta_0$ , and so the result is proved.  $\square$

**Corollary 4.3.** *All divisors interior to  $\mathbb{F}$  are semi-ample, and when  $\overline{M}_g$  is defined over  $\mathbb{C}$ , and  $g \leq 11$ , the divisor  $12\lambda - \delta_0$  on  $\overline{M}_g$  is semi-ample.*

*Proof.* By Theorem 4.2, the elements interior to  $\mathbb{F}$  are semi-ample. In [SB06], Shepherd-Barron shows that for  $g \leq 11$ , as long as  $\overline{M}_g$  is defined over  $\mathbb{C}$ , then  $12M - D_g^{vor}$ , which we show pulls back by  $f_{per}$  to  $12\lambda - \delta_0$ , is semi-ample.  $\square$

In [Rul01], it is shown that every nef divisor on  $\overline{M}_3$  is semi-ample, and so in particular,  $12\lambda - \delta_0$  was known to be semi-ample in that case.

## 5. THE DIVISORS THAT GIVE THE EXTENSIONS OF THE TORELLI MAP

In this section, we prove that the pullback of an ample divisor under any extension of the Torelli map lies on  $\mathbb{F}$ .

**Theorem 5.1.** *Let  $X$  be any compactification of  $A_g$  that contains Mumford's partial compactification  $A_g^{part}$  and maps to Satake's compactification  $A_g^{Sat}$ . Then if  $f : \overline{M}_g \rightarrow X$  is any extension of the Torelli map and  $A$  any ample divisor on  $X$ , there exists a constant  $c > 0$  and an  $\epsilon > 0$  for which  $f^*(cA) = (12 + \epsilon)\lambda - \delta_0$ .*

*Proof.* Let  $X$  be any compactification of  $A_g$  for which there is a morphism  $\eta : X \rightarrow A_g^{Sat}$ , and such that  $\eta^{-1}(A_g \cup A_{g-1}) = A_g^{part} = A_g \cup D_g$  is Mumford's partial compactification of  $A_g$ . Suppose that  $f : \overline{M}_g \rightarrow X$  any morphism that extends the Torelli map. In particular,  $f_{Sat} = \eta \circ f : \overline{M}_g \rightarrow A_g^{Sat}$  denotes the extension of the Torelli map from  $\overline{M}_g$  to the Satake compactification  $A_g^{Sat}$ .

We have the following commutative diagram.

$$\begin{array}{ccccc} & & X & & \\ & \nearrow f & \uparrow i & \searrow \eta & \\ \overline{M}_g & \xleftarrow{j} M_g^* & \xrightarrow{\phi} A_g^{part} & \xrightarrow{\quad} A_g^{Sat} = A_g \cup A_{g-1} \cup \cdots \cup A_0. \end{array}$$

Note that for  $M$ , the  $\mathbb{Q}$ -line bundle of modular forms of weight 1 on  $A_g^{Sat}$ , which is the ample generator of  $\text{Pic}(A_g^{Sat})$ , then  $f_{Sat}^*(M) = \lambda$  and that  $f_{Sat}^{-1}(A_g) = M_g^*$  is the moduli space of genus  $g$  stable curves of compact type [MFK94]. We denote by  $M$  the pull back of  $M$  to the three varieties  $M_g^*$ ,  $A_g^1$  and  $X$ .

The assumption that there is an embedding of  $A_g^{part}$  into  $X$  tells us that the Picard number of  $X$  is at least two, for it is known that  $\text{Pic}(A_g^{part}) \otimes \mathbb{Q} = \mathbb{Q}D_g \oplus \mathbb{Q}M$  (for  $g \in \{2, 3\}$ , see [vdG98], and for  $g \geq 4$ , see [Mum83, p. 355]). In particular, this means

that if  $A$  is any ample divisor on  $X$ , then  $f^*A$  cannot generate an extremal ray of the nef cone of  $\overline{M}_g$ , else the image of  $\overline{M}_g$  under  $f$  would have Picard rank 1.

The goal of this proof is to show that, if  $A$  is ample on  $X$ , then there exists an  $\epsilon > 0$  for which  $f^*A = \epsilon\lambda + 12\lambda - \delta_0$ , which is on the interior of  $\mathbb{F}$ . By Lemma 4.1, elements of the kernel of  $j^*$  are equivalent to a rational multiple of  $\delta_0$ . Let  $\{\lambda, \delta_0, B_3, \dots, B_d\}$  is any basis for  $\text{Pic}(\overline{M}_g)_{\mathbb{Q}}$ . Then  $f^*(A) = a\lambda - b\delta_0 - \sum_{i=3}^d b_i B_i$ , and  $j^*(f^*A) = aj^*\lambda - \sum_{i=3}^d b_i j^*B_i$ . We will first show that the  $b_i = 0$  for  $i \geq 3$ . To do this we'll see that  $\sum_{i=3}^d b_i j^*B_i \in \ker(j^*)$ .

Since  $A$  is ample on  $X$ , and  $i : A_g^{\text{part}} \hookrightarrow X$  is an embedding,  $i^*A$  is ample on  $A_g^{\text{part}}$ . In [HS04, Prop. I.7], it is proved that the nef cone of  $A_g^{\text{part}}$  is given by

$$\text{Nef}(A_g^{\text{part}}) = \{\alpha L - \beta D_g : \alpha \geq 12\beta \geq 0\}.$$

So we may write  $i^*A = \alpha M - \beta D_g$  such that  $\alpha \geq 12\beta \geq 0$ . But since  $M$  is nef but not ample, we also know that  $\beta > 0$ , so  $\alpha > 0$ . Moreover,  $\phi^*D_g = 0$ , since  $M_g^* = f_{\text{Sat}}^{-1}(A_g)$ . It follows therefore that  $j^*f^*(A) = \phi^*(i^*(A)) = \alpha L$ . Comparing the coefficients of the basis elements,

$$aj^*\lambda - \sum_{i=3}^d b_i j^*B_i = \alpha L - \sum_{i=3}^d b_i j^*B_i = \alpha L,$$

we see that  $a = \alpha$  and  $b_i = 0$ , for  $3 \leq i \leq d$ .

We therefore have that  $f^*(A) = a\lambda - b\delta_0$ , for  $b \in \mathbb{Q}_{\geq 0}$ . Moreover, since  $f^*A$  is a nef divisor on  $\overline{M}_g$ , it must nonnegatively intersect all  $F$ -curves and so  $a \geq 12b \geq 0$ . If  $b = 0$ , then  $f = a\lambda$ , which by Proposition 3 generates an extremal ray of the nef cone, and so the image of the map  $f : \overline{M}_g \rightarrow X$  has picard number 1, contradicting the assumption that  $A_g^{\text{part}}$  is embedded in  $X$ .

If  $a = 12b$ , then  $f^*(\frac{1}{b}A) = 12\lambda - \delta_0$ , which again by Proposition 3 generates an extremal ray of the nef cone, and so the image of the morphism  $f$  will be a variety of Picard rank 1, giving a contradiction.

We conclude that since  $a > 12b > 0$ , we have that  $f^*(\frac{1}{b}A) = \frac{a}{b}\lambda - \delta_0 = (12 + \epsilon)\lambda - \delta_0$ , for some  $\epsilon > 0$ . □

**Corollary 5.2.** *Suppose that  $f_{\tau} : \overline{M}_g \rightarrow A_g^{\tau}$  is any Toroidal extension of the Torelli map. Then the morphism  $f_{\tau}$  is given by a sub-linear series of divisor interior to  $\mathbb{F}$ .*

*Proof.* Suppose  $f_{\tau} : \overline{M}_g \rightarrow A_g^{\tau}$  is any Toroidal extension of the Torelli map. There is a morphism  $\eta_{\tau} : A_g^{\tau} \rightarrow A_g^{\text{Sat}}$  containing Mumford's partial compactification  $A_g^{\text{part}} = \eta_{\tau}^{-1}(A_g \cup A_{g-1})$ . So by Theorem 1.1, the result holds. □

**Remark 5.3.** The closure,  $\overline{H}_g$  in  $\overline{M}_g$  of the moduli space of hyperelliptic curves  $H_g \subset M_g$  is equal to the image of the morphism  $h : \overline{M}_{0,2(g+1)}/S_{2(g+1)} \rightarrow \overline{M}_g$  defined by taking  $(C, \bar{p}) \in \overline{M}_{0,2(g+1)}$  to the stable  $n$ -pointed curve of genus  $g$  obtained by taking a double cover of  $C$  branched at the (unordered) marked points. For  $\epsilon = \frac{1}{3 \cdot 2g - 5}$ , and  $c = \frac{1}{4}$ , one has that  $\kappa_1 = h^*(c(\epsilon\lambda + 12\lambda - \delta_0))$ . In particular, the morphism on  $\overline{M}_{0,2(g+1)}/S_{2(g+1)}$  given by  $\kappa_1$  factors through  $h$ .

**Remark 5.4.** In Corollary 4.3 it was shown that a multiple of  $12\lambda - \delta_0$  is base point free for  $g \leq 11$ . It is natural to look for the variety  $Y$  (which must necessarily have Picard number 1) and the corresponding morphism  $f_{m|12\lambda - \delta_0|} : \overline{M}_g \rightarrow Y$ .

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